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FREE VIBRATION RESPONSE OF SHEAR-DEFORMABLE ANTISYMMETRIC CROSS-PLY CYLINDRICAL PANELS

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A hitherto unavailable analytical solution to the boundary value problem of free vibration response of shear-flexible antisymmetric cross-ply laminated cylindrical panels is presented. The equivalent single layer approach based on a first order shear deformation theory including rotary and in-plane inertias is incorporated into the shell formulation. The characteristic equations of the panel are defined by five highly coupled second and third order partial differential equations in five unknowns, i.e., three displacements, and two rotations. A recently developed solution methodology, based on a boundary-continuous double Fourier series approach, is utilized to solve the eigenvalue problem. Numerical results presented for various parametric effects such as length-to-thickness ratio, radius-to-thickness ratio, aspect ratio, and major-to-minor modulus ratio, etc., should serve as a bench mark for future comparison. A four-node shear-flexible finite element is selected to compare the results with the present solution.

1. INTRODUCTION

Free vibration response studies of structural components farbicated with advanced fiber reinforced laminated composite materials such as graphite/epoxy, E-glass/epoxy, boron/epoxy, Kevlar-49/epoxy, etc., have received a great deal of attention in recent years. This study provides the characteristic properties of such components. These structural components, which are commonly in the form of plates, cylindrical panels, doubly curved panels or shells, etc., and used in aircraft fuselages, rocket motor cases, submarine fuselages, etc., play a great role in weight saving through use of composite materials possessing high strength-to-weight ratio and stiffness-to-weight ratio properties. These components require an aeroelastic tailoring which is essentially manipulation of structural responses, such as natural frequencies, divergence, and flutter speed, etc., for an optimized performance. Recent advancement in composites in the commercial aircraft sector, e.g., all-composite empennages on the Boeing 7J7 and Douglas MD-91X, is to limit sonic fatigue caused by the new fuel efficient propfan or unducted fan (UDF) engines. All this advancement and utilization demand more understanding, in terms of characteristic properties of

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such structural components. In this paper, the study is geared towards a general antisymmetric cross-ply laminated cylindrical panel.

The theoretical development in the field of linear elastic shell/panels has received a respected amount of consideration by numerous authors [1–14], in the past few decades, who have utilized various approximations with respect to the three-dimensional theory of curved deformable bodies. The above works have generated mainly three types of shell theories: Thin Shell Theory (TST), Moderately-Thick Shell Theory (MTST), and a more accurate moderately Thick Shell Theory (THST). The first shell theory (TST) completely ignores the effects of shear deformation, while the second one (MTST) considers a constant variation of it across the thickness. The last shell theory, THST, is based on a parabolic variation of transverse shear deformation, where transverse shear stress vanishes at the top and bottom surfaces of the shell thickness. The aforementioned three theories are known as equivalent single layer theories and are applied to laminated shells/panels/plates. For a spatial response, though the THST ranks highly, the use of MTST is popular because of its simplicity in formulation. Therefore, in this paper the MTST will be employed for developing a set of partial differential equations that characterizes the antisymmetric cross-ply cylindrical panel behavior in dynamic response.

In order to consider the effects of curvature in the shell formulations various theories were developed, namely, Donnell [15], Love [1], Flugge [11], Sanders [7], Donnell–Mushtari–Vlasov [9] shell theories, etc., are popular among them. In this paper consideration will be given to the Sanders [7] curvature-based shell theory. The aforementioned shell theories were initially developed for isotropic materials, and later extended for laminated composite materials [16–18].

The efforts of development as seen for the case of shell theories are not as conspicuous as for the case of development of analytical techniques for solving boundary value problems of these shells. These are, in part, due to the rapid growth experienced by such popular approximate numerical methods, e.g., finite difference and finite element methods, but more significantly due to the formidable difficulties posed by the system of highly coupled partial differential equations that arise from the in-plane bending, bending–twisting coupling effects due to the lamination sequence, and stacking pattern, and most importantly fulfilling the necessary conditions associated with prescribed admissible boundary conditions. The intention of the present study is to develop a suitable analytical solution, for a free vibration response of a cylindrical panel with antisymmetric cross-ply lamination.

The majority of the studies on cross-ply shells and panels utilized either TST or MTST. Stavsky and Lowey [19], Jones and Morgan [20], and Greenberg and Stavsky [21] have all utilized the TST in obtaining analytical solutions to the vibration and buckling problems of cross-ply cylindrical shells. The TST-based analytical solutions for vibration and buckling response of cross-ply cylindrical panels were presented by Soldatos and Tzivandis [22]. Jones and Morgan [20] and Soldatos and Tzivandis [22] have used Donnell's [15] kinematic relation, while Stavsky and Lowey [19] and Greenberg and Stavsky [21] have utilized a Love [1] type theory. However, in another work Soldatos [17] has discussed four well-known shell theories, Donnell [15], Love [1], Sanders [7], and Flugge [11], using the approximate Galerkin's approach. Soldatos [23] has also presented the solution of vibration problem of cross-ply cylindrical shells utilizing the second approximation of Flugge-type theory. Dong, Pister and Taylor [24] have developed a theory for anisotropic thin shells, employing Donnell's [15] shell theory, and presented only solutions for cross-ply laminates.

Gulati and Essenberg [25] and Zukas and Vinson [26] have introduced MTST in studying the behavior of complete cylindrical shells with isotropic materials. Dong and Tso [14] have obtained analytical solution to free vibration problems of cross-ply shells, based on MTST. Sinha and Rath [27], using MTST, have presented exact solutions, for circular cylindrical panels with cross-ply lamination. They have used Donnell's kinematic relation in the shell formulations. Bert and Kumar [18] have successfully obtained MTST-based exact solutions vibration problems. Reddy [16] using Sanders kinematic relations for MTST has presented an analytical solution to cross-ply doubly curved shells and cylindrical panels. The solution approach adopted by Bert and Kumar [18] and Reddy [16] was based on the Navier's theory of double Fourier series method. Recently, Librescu *et al*. [28] and Khdeir *et al*. [29] have reported analytical solutions to various boundary conditions for cross-ply cylindrical panels with MTST. The solution methodology that they have used is according to Levy's approach. The above mentioned two approaches—Navier and Levy types—can handle only limited number of boundary conditions. As for example, boundary conditions with (1) displacement restrained along transverse direction and normal to the support, and (2) rotations restrained about the normal to the support, may not be easily handled by the above methods. As a matter of fact this boundary condition with MTST to the vibration response of antisymmetric cross-ply lamination is yet to be addressed analytically, a gap in the analytical study still exists.

Therefore, the purpose of the present investigation is to develop an analytical solution scheme for the aforementioned boundary conditions, for the dynamic response of a cylindrical panel with antisymmetric cross-ply laminations.

2. BASIC EQUATIONS

A laminated cylindrical panel of total thickness, h, is shown in Figure 1. The thickness of the kth layer is denoted by $t^{(k)} = \alpha_3^k - \alpha_3^{k-1}$, in which $\alpha_3^{(k)}$ and $\alpha_3^{(k-1)}$, $k = 1, 2, \ldots, N$, are the distances from the reference surface to the outer and inner faces, respectively, of the layer measuring away from the mid-depth of the panel. An orthogonal curvilinear co-ordinate system is selected to define the geometry of the panel. The co-ordinate system is placed at the mid height of the panel thickness. The curvilinear axes $\alpha_1\alpha_2$ define the reference surface ($\alpha_3 = 0$) to the panel. The radius R is measured to the reference surface. The span a is measured along the axis α_1, while the span b is along the α_2-axis. The equations of motion, based on the Sanders [7] moderately deep shell theory, can be written as

$$\frac{\partial N_1}{\partial \alpha_1} + \frac{\partial N_6}{\partial \alpha_2} + \frac{1}{2R}\frac{\partial M_6}{\partial \alpha_2} + \frac{Q_1}{R} = m_1, \qquad \frac{\partial N_6}{\partial \alpha_1} - \frac{1}{2R}\frac{\partial M_6}{\partial \alpha_1} + \frac{\partial N_2}{\partial \alpha_2} = m_2,$$

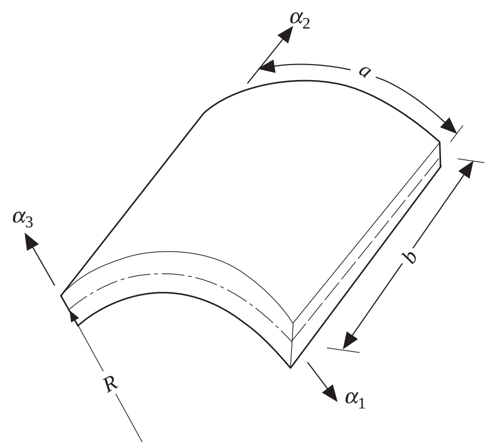

Figure 1. A cylindrical panel.

$$\frac{\partial Q_1}{\partial \alpha_1} + \frac{\partial Q_2}{\partial \alpha_2} - \frac{N_1}{R} = m_3, \qquad \frac{\partial M_1}{\partial \alpha_1} + \frac{\partial M_6}{\partial \alpha_2} - Q_1 = m_4,$$

$$\frac{\partial M_6}{\partial \alpha_1} + \frac{\partial M_2}{\partial \alpha_2} - Q_2 = m_5, \tag{1a–e}$$

where N_1, N_2, and N_6 are the surface parallel stress resultants, while M_1, M_2, and M_6 are stress couple resultants, and Q_1 and Q_2 are the transverse shear stress resultants, all per unit length. m_i $(i = 1, \ldots, 5)$ are defined as

$$m_1 = \left(\rho_1 + \frac{2\rho_2}{R}\right)\ddot{u}_1 + \left(\rho_2 + \frac{2\rho_3}{R}\right)\ddot{\phi}_1, \qquad m_2 = \rho_1 \ddot{u}_2 + \rho_2 \ddot{\phi}_2, \qquad m_3 = \rho_1 \ddot{u}_3,$$

$$m_4 = \left(\rho_2 + \frac{\rho_3}{R}\right)\ddot{u}_1 + \rho_3 \ddot{\phi}_1, \qquad m_5 = \rho_2 \ddot{u}_2 + \rho_3 \ddot{\phi}_2, \tag{2a–e}$$

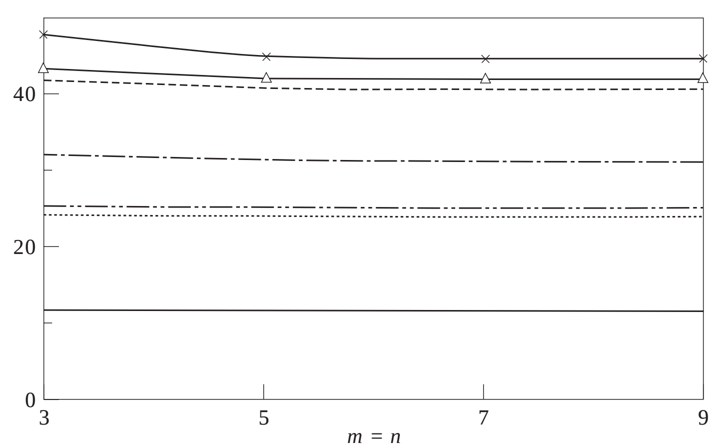

Figure 2. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 1$, $R/a = 10$ and $a/h = 10$. Key: —, λ_1; · · · ·, λ_2; – · · –, λ_3; – · –, λ_4; - - -, λ_5; –△–, λ_6; –×–, λ_7.

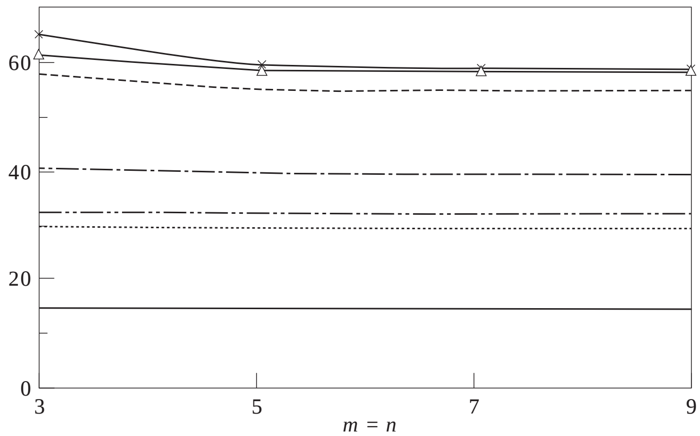

Figure 3. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 1$, $R/a = 10$ and $a/h = 20$. Key as for Figure 2.

in which surface-parallel and rotatory inertias are included. u_i ($i = 1, 2, 3$) represent the displacement components of the reference surface along α_i ($i = 1, 2, 3$)-axes, respectively. ϕ_i ($i = 1, 2$) represent rotations of normal about α_i ($i = 1, 2$)-axes, respectively. $(\ddot{\ })$ represents second derivative with respect to time. ρ_i ($i = 1, 2, 3$) are defined as

$$(\rho_1, \rho_2, \rho_3) = \sum_{K=1}^{N} \int_{\alpha_3^{(k-1)}}^{\alpha_3^{(k)}} \rho^{(k)}(1, \alpha_3, \alpha_3^2)\, \mathrm{d}\alpha_3, \tag{3}$$

where $\rho^{(k)}$ and N represent the density of the layer material, and the total number of layers, respectively. For a general cross-ply laminated panel, surface-parallel stress resultants, N_i, stress couples, M_i, and transverse shear stress resultants, Q_i,

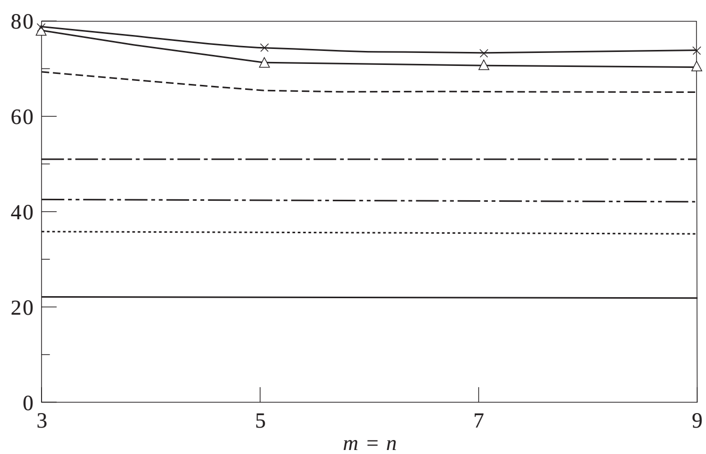

Figure 4. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 1$, $R/a = 10$ and $a/h = 50$. Key as for Figure 2.

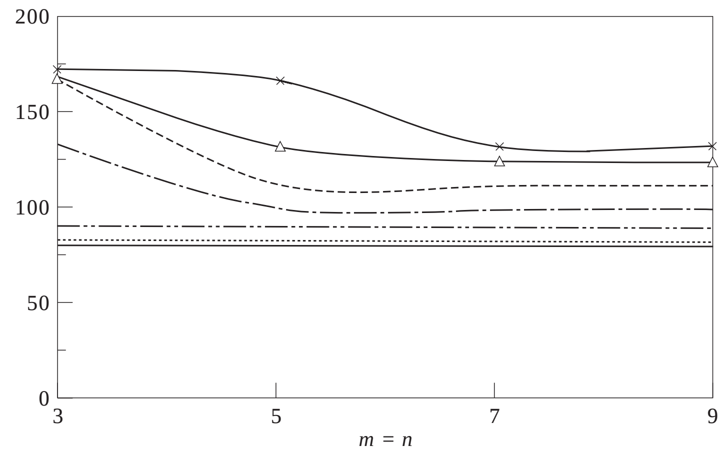

Figure 5. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 5$, $R/a = 10$ and $a/h = 10$.

are related to the mid-surface strains, ϵ_i^0, and changes of curvature and twist, κ_i, by

$$N_i = A_{ij}\epsilon_j^0 + B_{ij}\kappa_i \quad (i, j = 1, 2), \qquad N_6 = A_{66}\epsilon_6^0 + B_{66}\kappa_6,$$

$$M_i = B_{ij}\epsilon_j^0 + D_{ij}\kappa_j \quad (i = j = 1, 2), \qquad M_6 = B_{66}\epsilon_6^0 + D_{66}\kappa_6,$$

$$Q_1 = A_{55}\epsilon_5^0 \qquad Q_2 = A_{44}\epsilon_4^0, \qquad A_{55} = K_1^2 A_{55}, \qquad A_{44} = K_2^2 A_{44}, \qquad \text{(4a–h)}$$

where A_{ij}, B_{ij}, and D_{ij} [30] are extensional, coupling, and bending rigidities, respectively. A_{44} and A_{55} [30] represent transverse shear rigidities. K_1^2 and K_2^2 are shear correction factors. ϵ_j^0 $(j = 1, 2, 4, 5, 6)$ and k_j $(j = 1, 2, 6)$, related displacement functions and their derivatives, are as defined in reference [31], and not presented here for the sake of brevity.

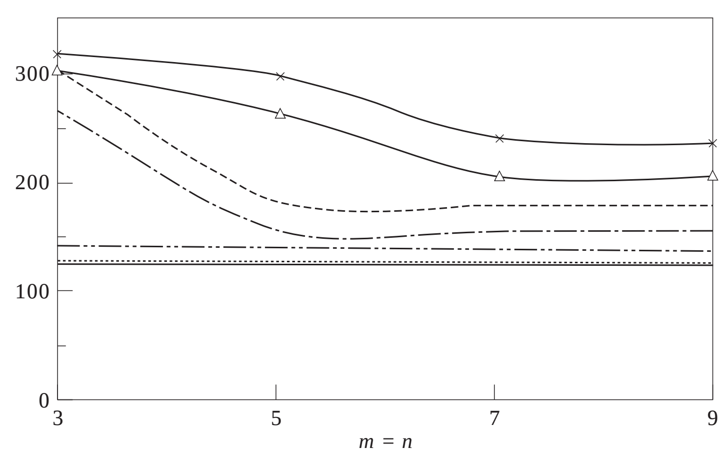

Figure 6. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 5$, $R/a = 10$ and $a/h = 20$.

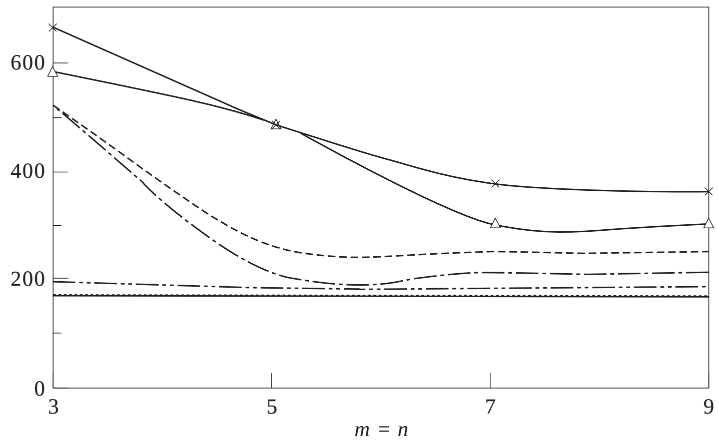

Figure 7. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 5$, $R/a = 10$ and $a/h = 50$. Key as for Figure 2.

Introduction of equations (4) into equations (1) yields five highly coupled partial differential equations with constant coefficients. These equations can be written in a matrix form as:

$$\mathscr{L}\mathbf{v} = \mathbf{f}, \tag{5}$$

where

$$\mathscr{L}_{ij} = \mathscr{L}_{ji}, \qquad i, j = 1, \ldots, 5, \qquad \mathbf{v}^{\mathrm{T}} = \{u_1, u_2, u_3, \phi_1, \phi_2\},$$
$$\mathbf{f}^{\mathrm{T}} = \{m_1, m_2, m_3, m_4, m_5\}. \tag{6a–c}$$

The operators $\mathscr{L}_{ij}$ can be written as follows:

$$\mathscr{L}_{11} = -\frac{A_{55}}{R^2}(\,) + A_{12}(\,)_{,\alpha_1\alpha_1} + \left(A_{66} + \frac{2}{R}B_{66} + \frac{1}{R^2}D_{66}\right)(\,)_{,\alpha_2\alpha_2}$$

$$\mathscr{L}_{12} = \left(A_{12} + A_{66} - \frac{1}{R^2}\right)(\,)_{,\alpha_1\alpha_2}, \qquad \mathscr{L}_{13} = \left(\frac{A_{11}}{R} + \frac{A_{55}}{R}\right)(\,)_{,\alpha_1\alpha_2},$$

$$\mathscr{L}_{14} = -\frac{A_{55}}{R}(\,) + B_{11}(\,)_{,\alpha_1\alpha_1} + \left(B_{66} + \frac{1}{R}D_{66}\right)(\,)_{,\alpha_2\alpha_2},$$

$$\mathscr{L}_{15} = \left(B_{12} + B_{66} - \frac{1}{R}D_{66}\right)(\,)_{,\alpha_1\alpha_2}, \qquad \mathscr{L}_{22} = \left(A_{66} - \frac{2}{R}B_{66} + \frac{1}{R^2}D_{66}\right)(\,)_{,\alpha_1\alpha_2},$$

$$\mathscr{L}_{23} = \frac{A_{12}}{R}(\,)_{,\alpha_2}, \qquad \mathscr{L}_{24} = \left(B_{12} + B_{66} - \frac{1}{R}D_{66}\right)(\,)_{,\alpha_1\alpha_2},$$

$$\mathscr{L}_{25} = \left(B_{66} - \frac{1}{R}D_{66}\right)(\,)_{,\alpha_1\alpha_2},$$

$$\mathscr{L}_{33} = -\frac{A_{11}}{R^2}(\,) + A_{55}(\,)_{,\alpha_1\alpha_1} + A_{44}(\,)_{,\alpha_2\alpha_2}, \qquad \mathscr{L}_{34} = \left(A_{55} - \frac{B_{11}}{R}\right)(\,)_{,\alpha_1},$$

$$\mathscr{L}_{35} = \left(A_{44} - \frac{B_{12}}{R}\right)(\,)_{,\alpha_2}, \qquad \mathscr{L}_{44} = -A_{55}(\,) + D_{11}(\,)_{,\alpha_1\alpha_1} + D_{66}(\,)_{,\alpha_2\alpha_2},$$

$$\mathscr{L}_{45} = -(D_{12} + D_{66})(\,)_{,\alpha_1\alpha_2},$$

$$\mathscr{L}_{55} = -A_{44}(\,) + D_{66}(\,)_{,\alpha_1\alpha_1} + D_{22}(\,)_{,\alpha_2\alpha_2}. \tag{7a–o}$$

Admissible boundary conditions of the following form are chosen:

$$u_1(0, x_2) = u_1(a, x_2) = u_2(x_1, 0) = u_2(x_1, b) = 0,$$

$$u_3(0, x_2) = u_3(a, x_2) = u_3(x_1, 0) = 3_2(x_1, b) = 0,$$

$$N_6(0, x_2) = N_6(a, x_2) = N_6(x_1, 0) = N_6(x_1, b) = 0,$$

$$M_1(0, x_2) = M_1(a, x_2) = M_2(x_1, 0) = M_2(x_1, b) = 0. \tag{8a–d}$$

The main objective here is to solve equation (5) in conjunction with equation (8).

3. SOLUTION TO THE BOUNDARY VALUE PROBLEM

The assumed solution functions for the finite dimensional cross-ply laminated cylindrical panel boundary value problem are selected in terms of double Fourier series in the following form [31]:

$$u_1(\alpha_1, \alpha_2, t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} C^{\mathrm{I}}_{mn} \sin\left(\frac{m\pi\alpha_1}{a}\right)\cos\left(\frac{n\pi\alpha_2}{b}\right) \mathrm{e}^{\mathrm{i}\omega t},$$

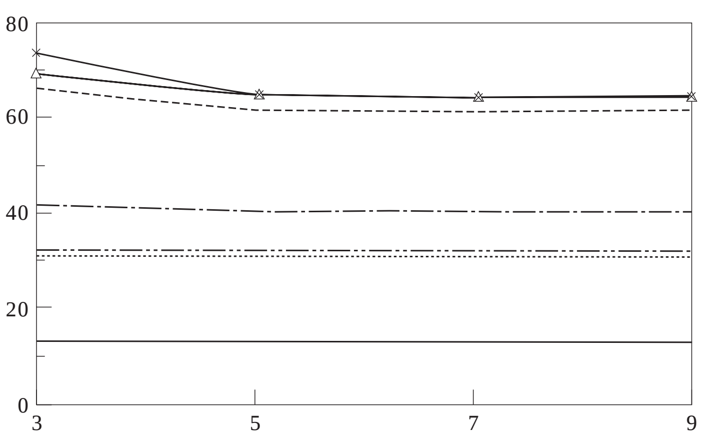

Figure 8. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 1$, $R/a = 50$ and $a/h = 50$.

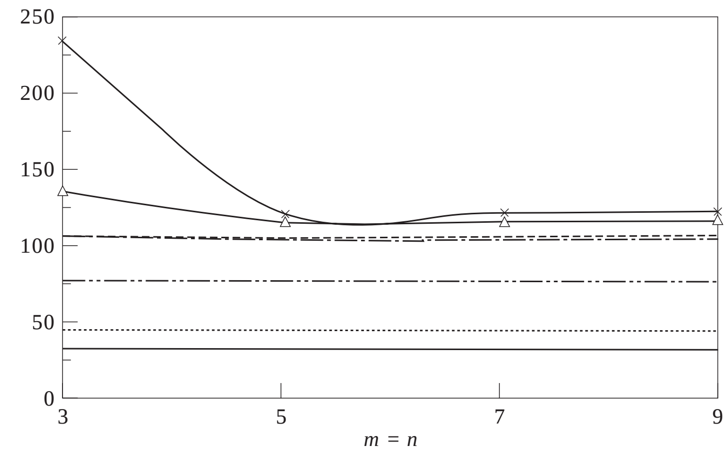

Figure 9. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 2$, $R/a = 50$ and $a/h = 50$.

$$u_2(\alpha_1, \alpha_2, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn}^{\mathrm{II}} \cos\left(\frac{m\pi\alpha_1}{a}\right) \sin\left(\frac{n\pi\alpha_2}{b}\right) \mathrm{e}^{\mathrm{i}\omega t},$$

$$u_3(\alpha_1, \alpha_2, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn}^{III} \sin\left(\frac{m\pi\alpha_1}{a}\right) \sin\left(\frac{n\pi\alpha_2}{b}\right) \mathrm{e}^{\mathrm{i}\omega t},$$

$$\phi_1(\alpha_1, \alpha_2, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn}^{\mathrm{IV}} \cos\left(\frac{m\pi\alpha_1}{a}\right) \cos\left(\frac{n\pi\alpha_2}{b}\right) \mathrm{e}^{\mathrm{i}\omega t},$$

$$\phi_2(\alpha_1, \alpha_2, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn}^{\mathrm{V}} \sin\left(\frac{m\pi\alpha_1}{a}\right) \cos\left(\frac{n\pi\alpha_2}{b}\right) \mathrm{e}^{\mathrm{i}\omega t}, \quad \text{(9a–e)}$$

where C_{mn}^{i} $(i = \mathrm{I}, \ldots, \mathrm{V})$ are Fourier constants; i is defined as $\sqrt{-1}$; and ω indicates an eigenvalue.

It is worth mentioning here that the above solution functions are successful for the case of antisymmetric angle ply laminated cylindrical panels where the Navier's approach has been utilized [16], and these functions have never been applied for the use of cross-ply laminated panels. The above solution functions completely satisfy the boundary conditions as prescribed by u_n, u_3, and ϕ_t at the respective edges. Therefore, their first derivatives may be obtained without any difficulties. As the governing partial differential equations contain second derivatives, therefore, it is a necessary condition that the first derivative be derivable. An example can be given with $(\,)_{,\alpha_1\alpha_1}$, taking the displacement function of u_2 as it appears in the term $\mathscr{L}_{11}$:

$$\frac{\partial u_2}{\partial \alpha_1} = -\sum_{m=1} \sum_{n=1} C_{mn}^{\mathrm{II}} \frac{m\pi}{a} \sin\left(\frac{m\pi\alpha_1}{a}\right) \sin\left(\frac{n\pi\alpha_2}{b'}\right) \mathrm{e}^{\mathrm{i}\omega t}, \quad 0 < \alpha_1 < a, \quad 0 \leqslant \alpha_2 \leqslant b. \quad (10)$$

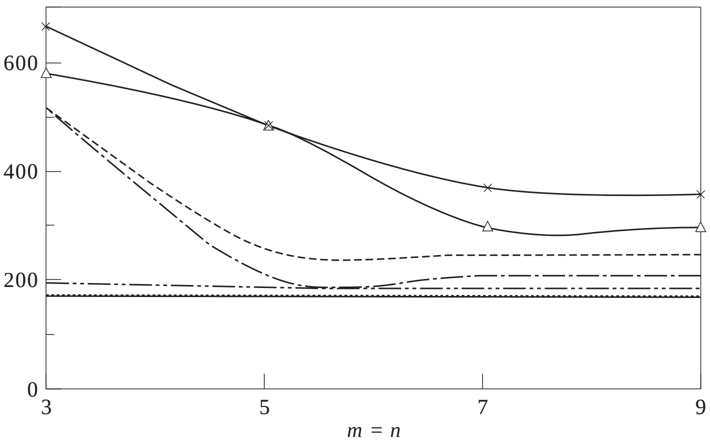

Figure 10. Convergence characteristics of natural frequencies (λ_1–λ_7) of a cylindrical panel with $a/b = 5$, $R/a = 50$ and $a/h = 50$.

The further derivative of it with respect to α_1 cannot be performed in an ordinary sense, as it shows discontinuities at the edges $\alpha_1 = 0$ and a. In such a situation $\partial u_2/\partial\alpha_1$ is then expanded into the form as prescribed by Hobson [32], Goldstein [33], Green [34], Green and Hearmon [35], Whitney [36], Whitney and Leissa [37], Chaudhuri [38], Kabir [31, 39], Chaudhuri and Kabir [40–43], and Kabir and Chaudhuri [44] in the following form:

$$u_{2,11}(x_1, x_2, t) = \left\{\frac{1}{2}\sum_{n=1}^{\infty} C_{on}^{\mathrm{II}} \sin\left(\frac{m\pi\alpha_1}{a}\right)\right.$$

$$\left. + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-\frac{m^2\pi^2}{b^2} C_{mn}^{\mathrm{II}} + \langle 0, 1\rangle \beta_n^{\mathrm{I}} + \langle 1, 0\rangle \beta_n^{\mathrm{II}}\right] \cos\left(\frac{m\pi\alpha_1}{a}\right) \sin\left(\frac{n\pi\alpha_2}{b}\right)\right\} \mathrm{e}^{\mathrm{i}\omega t},$$

$$0 \leqslant \alpha_1 \leqslant a, \qquad 0 \leqslant \alpha_2 \leqslant b, \quad (11)$$

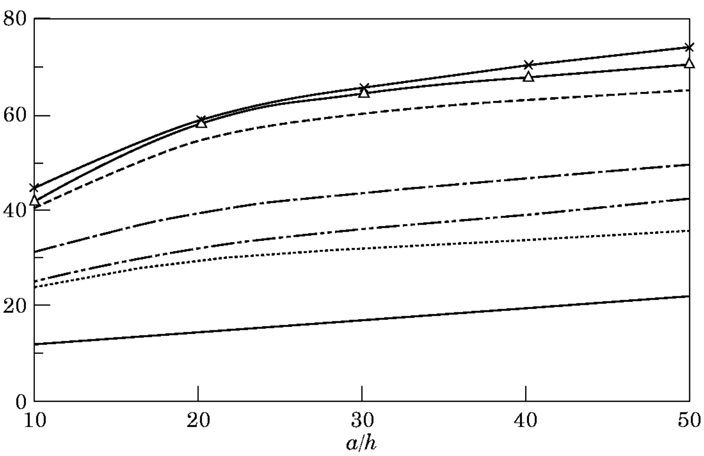

Figure 11. Variations of normalized lowest seven eigenvalues for various a/h for a cylindrical panel with $a/b = 1$, $R/a = 10$ and $m = n = 7$.

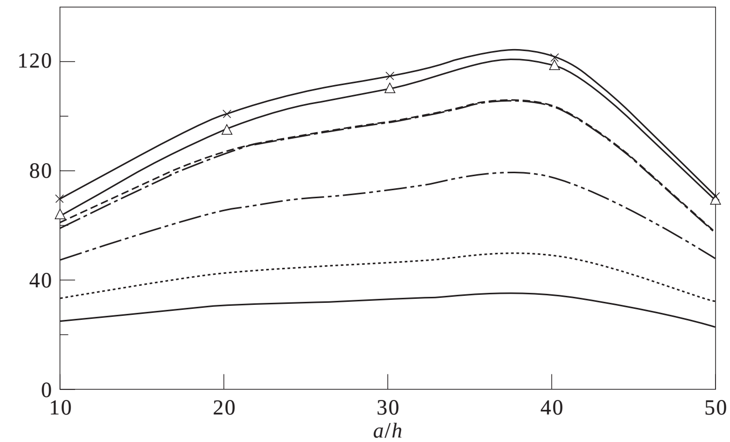

Figure 12. Variations of normalized lowest seven eigenvalues for various a/h for a cylindrical panel with $a/b = 2$, $R/a = 10$ and $m = n = 7$.

where $\langle 0, 1\rangle$ equals 1 if m or n is even, and zero if m or n is odd. $\langle 1, 0\rangle$ indicates the reverse of $\langle 0, 1\rangle$. The unknowns, β_n^{I} and β_n^{II}, can be related to $u_{2,1}$ obtained at the boundaries in the following form:

$$u_{2,1}(0, x_2) = -\frac{a}{4}\sum_{n=1}^{\infty}(\beta_n^{\mathrm{I}} + \beta_n^{\mathrm{II}})\sin\left(\frac{n\pi\alpha_2}{b}\right)\mathrm{e}^{\mathrm{i}\omega t},$$

$$u_{2,1}(a, x_2) = +\frac{a}{4}\sum_{n=1}^{\infty}(\beta_n^{\mathrm{I}} - \beta_n^{\mathrm{II}})\sin\left(\frac{n\pi\alpha_2}{b}\right)\mathrm{e}^{\mathrm{i}\omega t}. \quad (12)$$

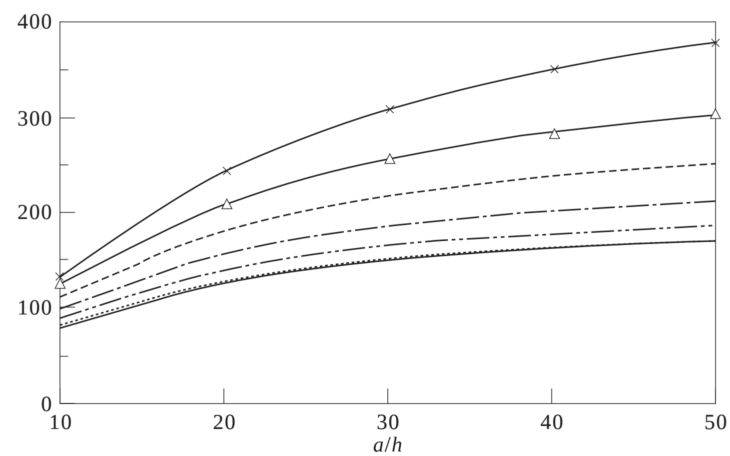

Figure 13. Variations of normalized lowest seven eigenvalues for various a/h for a cylindrical panel with $a/b = 5$, $R/a = 10$ and $m = n = 7$.

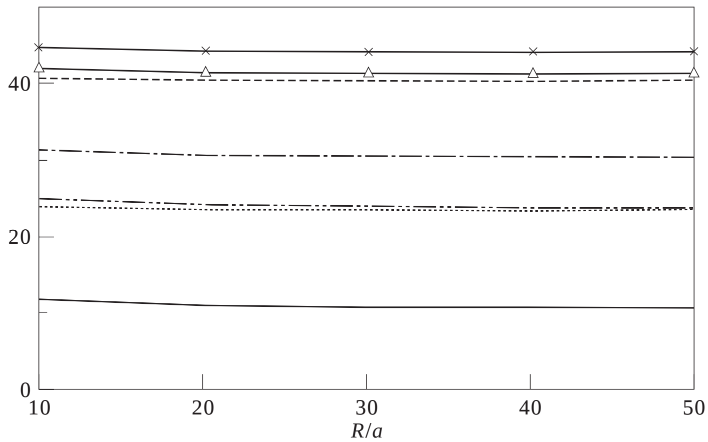

Figure 14. Variations of normalized lowest seven eigenvalues for various R/a for a cylindrical panel with $a/b = 1$, $a/h = 10$ and $m = n = 7$.

The similar applications are performed wherever applicable. Finally, the differential equations (5) together with the boundary conditions (8) will generate the necessary unknowns equal to number of equations.

A computer program AFSANA-VIB (A Fourier Series ANAlysis-VIBration) has been developed using FORTRAN code on a SUN workstation. The eigenvalues and mode shapes are computed calling the commercial software IMSL [45] as a subroutine.

4. NUMERICAL RESULTS AND DISCUSSIONS

In what follows, numerical results for natural frequencies and mode shapes of cylindrical panels with various parametric effects are presented. The following material properties of a graphite/epoxy lamina are considered: $E_1 = 76\text{E}3$ GPa,

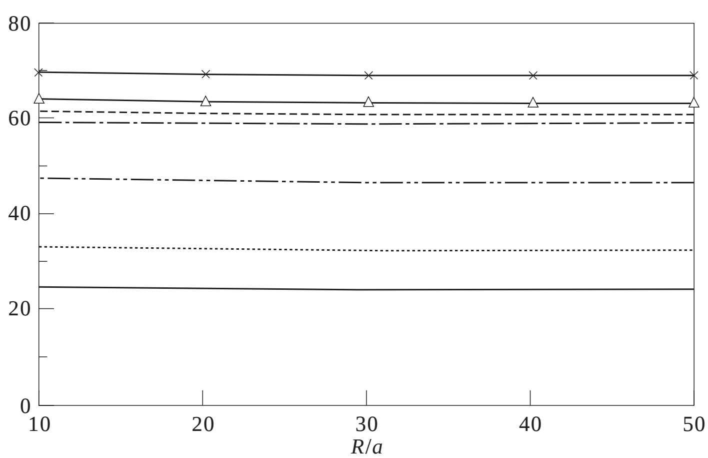

Figure 15. Variations of normalized lowest seven eigenvalues for various R/a for a cylindrical panel with $a/b = 2$, $a/h = 10$ and $m = n = 7$.

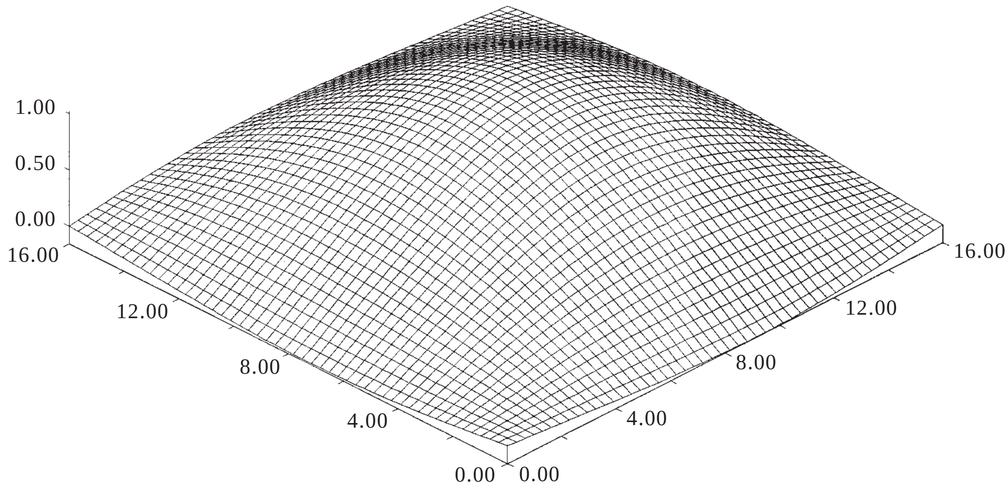

Figure 16. Mode shape, $u_{3(1,1)}$ for a cylindrical panel with $a/b = 1$, $R/a = 10$ and $a/h = 10$.

$E_2 = 5{\cdot}5\text{E}3$ GPa, $\nu_{12} = 0{\cdot}34$, $E_1\nu_{21} = E_2\nu_{21}$, $G_{13} = 2{\cdot}5\text{E}3$ GPa, $G_{12} = 1{\cdot}5\text{E}3$ GPa, $G_{23} = 2{\cdot}5\text{E}3$ GPa, $K_1^2 = K_2^2 = 5/6$, where E_1 and E_2 are the major and minor surface parallel Young's moduli along and across the fiber direction of a lamina, respectively. ν_{12} denotes the major Poisson's ratio. G_{13}, G_{12}, and G_{23} are shear moduli. For the sake of convenience, the natural frequencies are presented in a normalized form. The following is the relation for normalization: $\lambda_i = \overline{\lambda}_i a^2 (\sqrt{\rho/E_2})/h$, $i = 1, 2, \ldots$, where $\overline{\lambda}_i$ represents the ith lowest natural frequency, and λ_i its corresponding normalized one. A total of seven lowest natural frequencies are computed. A convergence study of the first seven lowest frequencies for a cylindrical panel with $a/b = 1$ and $R/a = 10$, and various a/h ($=10, 20, 50$) is

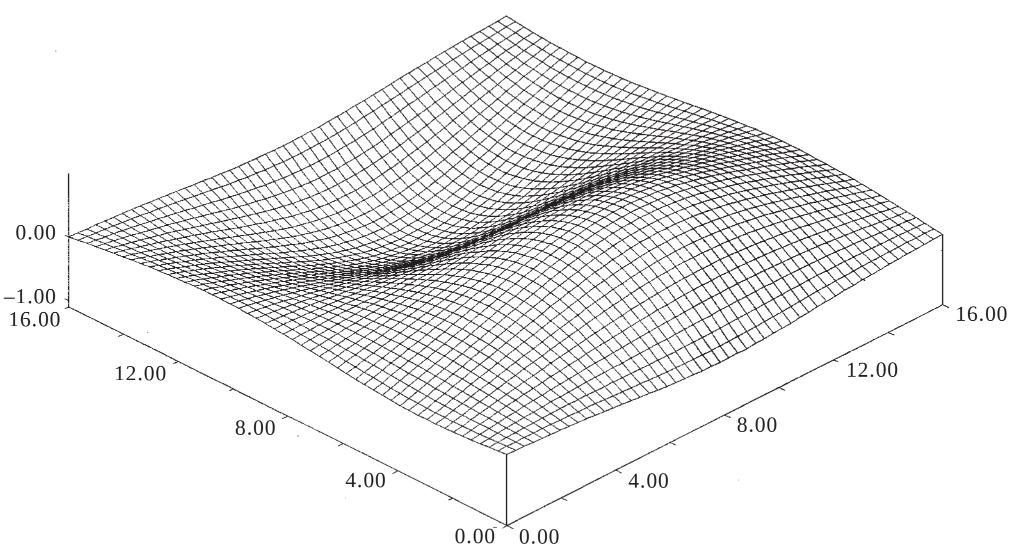

Figure 17. Mode shape, $u_{3(1,2)}$ for a cylindrical panel with $a/b = 1$, $R/a = 10$ and $a/h = 10$.

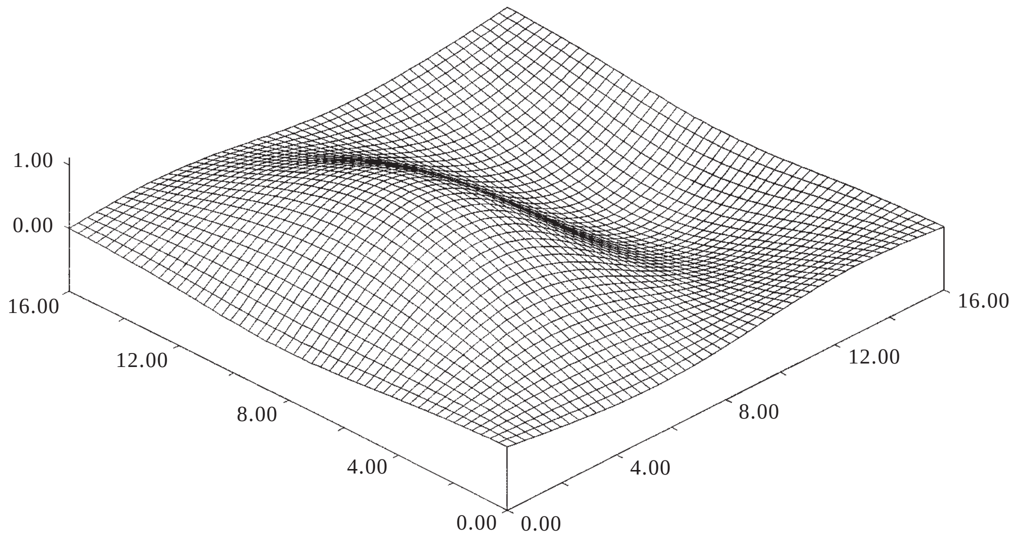

Figure 18. Mode shape, $u_{3(1,3)}$ for a cylindrical panel with $a/b = 1$, $R/a = 10$ and $a/h = 10$.

presented in Figures 2–4, respectively. For all a/h, λ_i ($i = 1$–7) converge satisfactorily with the increase of the number of terms in the Fourier series. Figures 5–7 illustrate convergence of λ_i ($i = 1$–7) for $a/b = 5$, $R/a = 10$, and $a/h = 10$, 20 and 50, respectively. The normalized λ_i ($i = 1$–3) converge very convincingly, while for λ_i (4–7) high oscillations are noticed for $m = n < 7$, and they diminish with $m = n > 7$. Figures 8–10 plot the convergence of λ_i ($i = 1$–7) for a cylindrical panel with $R/a = 50$, $a/h = 50$, and $a/b = 1$, 2, and 5, respectively. All of them show satisfactory convergence with $m = n > 7$. Variations of the eigenvalues (λ_i, $i = 1$–7) for a panel with $a/b = 1$, $R/a = 10$, and $m = n = 7$ for

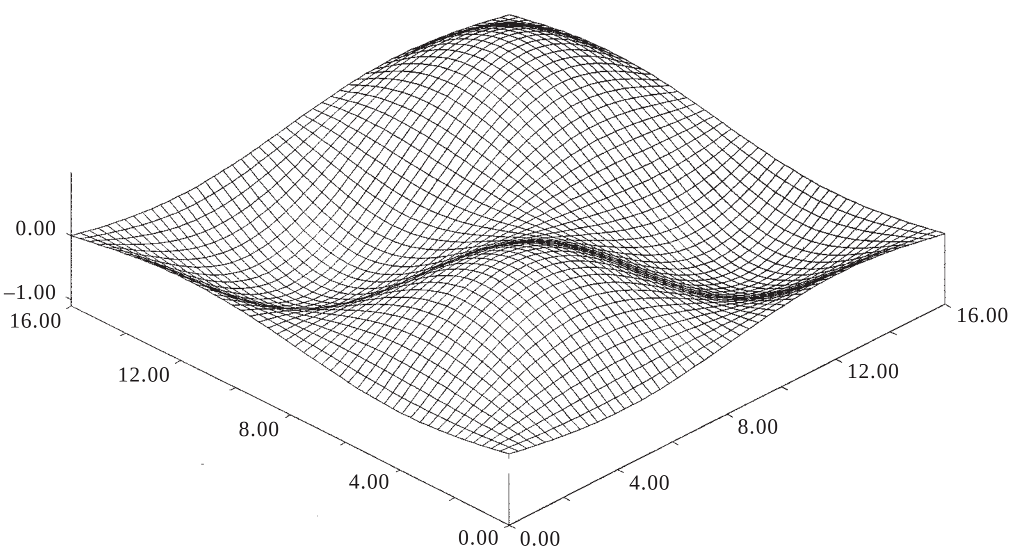

Figure 19. Mode shape, $u_{3(2,1)}$ for a cylindrical panel with $a/b = 1$, $R/a = 10$ and $a/h = 10$.

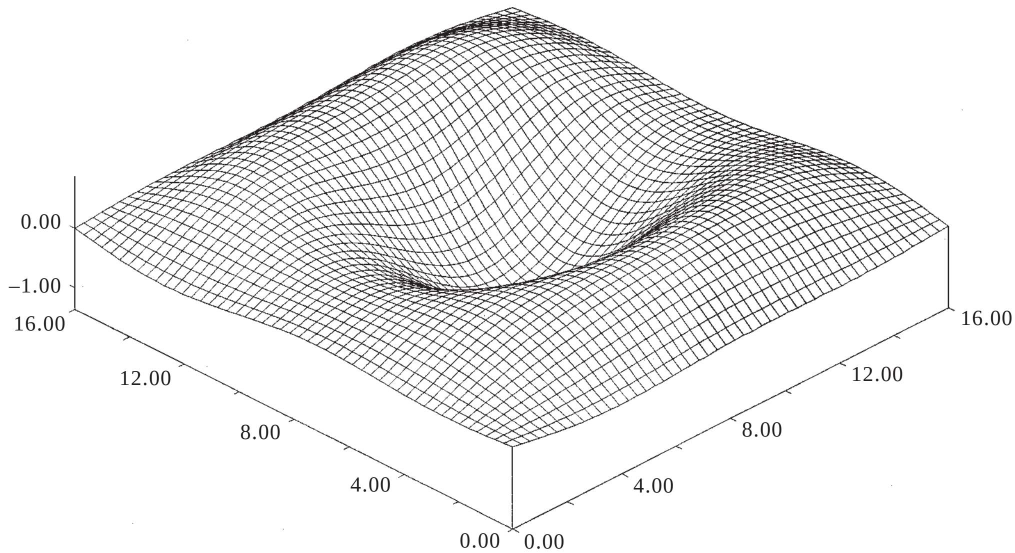

Figure 20. Mode shape, $u_{3(2,2)}$ for a cylindrical panel with $a/b = 1$, $R/a = 10$ and $a/h = 10$.

various a/h are elucidated in Figure 11. A natural frequency increases with an increment of a/h, a feature which is more prominent in the higher frequencies than in the lower ones. The same situations are not achieved for the case of the seam cylindrical panel but with $a/b = 2$ (Figure 12). The natural frequencies increase until $a/h = 40$, then start decreasing. This is more conspicuous in the higher frequencies. Figure 13 illustrates the natural frequencies (λ_i, $i = 1$–7) for a cylindrical panel with $a/b = 5$; $R/A = 10$, and $m = n = 7$. The frequencies increase with the increase of a/h. Figures 14 and 15 plot frequencies (λ_i, $i = 1$–7) for a panel with $a/b = 1$, and $a/b = 2$, respectively, for various radius-to-span ratios. The frequencies remain fairly constant for the range studied. Various mode shapes are plotted in Figures 16–20. In $u_3(i, j)$, (i, j) indicates mode shape associated with u_3.

TABLE 1

Comparison of present solution with available finite element [42] *results for a cylindrical panel with* $b/a = 1$, $R/a = 10$, *and* $a/h = 10$ *and* 50

	$a/h = 10$		$a/h = 50$	
Eigenvalues	Analytical	NISA [42]	Analytical	NISA [42]
λ_1	11·00	10·78	18·60	18·15
λ_2	25·40	19·84	31·50	25·00
λ_3	26·90	20·19	37·80	28·00
λ_4	31·40	26·50	44·30	35·00
λ_5	41·00	37·00	58·50	53·00
λ_6	42·28	37·30	64·00	54·00
λ_7	46·42	41·00	66·42	60·00

A comparison of the results obtained from a finite element analysis is presented in Table 1. The element that is considered is a four-node shear flexible one. The detail of the element formulation is available in Reference [46], omitted here for the sake of brevity. The lowest natural frequency obtained by both the methods are in very close agreement for $a/h = 10$ and 50. However, the same is not found for the higher frequencies. The finite element method under predicts the natural frequencies in all other cases. The cylindrical panel modelled with 32×32 finite elements, shows fairly converged results.

5. CONCLUSION

An analytical solution to a cross-ply laminated cylindrical panel is presented. The solution methodology adopted here is based on a boundary continuous solution functions approach. The numerical results are compared to a commercially available finite element package. The results presented should serve as bench-mark for approximately obtained solution methods. The continuation of this approach to buckling problems is currently underway.

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